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POSITIVE INTEGER SOLUTIONS OF THE DIOPHANTINE EQUATIONS $x^2-5F_nxy-5(-1)^ny^2=\pm 5^r$

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Abstract. In this study, we consider the Diophantine equations given in the title and determine when these equations have positive integer solutions. Moreover, we find all positive integer solutions of them in terms of Fibonacci and Lucas numbers.

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1. INTRODUCTION

The problem of determining all integer solutions to Diophantine equations has gained a considerable amount of interest among the mathematicians and there is a very broad literature on this subject. In particular, several papers [3–6, 10, 11, 16, 21, 22, 26] deal with the Diophantine equation

$$ax^2+bxy+cy^2=d,$$

where a,b,c, and d are fixed integers. The name "Diophantine" comes from Diophantus, an Alexandrian mathematician of the third century A. D., who proposed many Diophantine problems; but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. The main problem when studying a given Diophantine equation is that whether a solution exists, and, in the case it exists, how many solutions there are. Moreover, another important problem closely related to the previous one is that whether there is a general form for the solutions. Further details on Diophantine equations can be found in [7, 8, 17–19, 23, 24].

In [12], the authors considered the Diophantine equations

$$x^2-L_nxy+(-1)^ny^2=\pm 5^r$$

under the assumptions that $n>0$, $r>1$. They determined when the Diophantine equations $x^2-L_nxy+(-1)^ny^2=\pm 5^r$ have positive integer solutions. Moreover, they found all positive integer solutions of these equations in terms of Fibonacci and Lucas numbers. Based on the above equations, in this study, we consider the following Diophantine equations:

$$x^2-5F_nxy-5(-1)^ny^2=5^r$$

and

$$x^2-5F_nxy-5(-1)^ny^2=-5^r$$

with $n>1$, $r\geq 0$. The purpose of this study is to determine when the above equations have positive integer solutions and then find all positive integer solutions of them by using some properties of Fibonacci and Lucas sequences. As a reminder, for $n\geq 2$, the well-known Fibonacci and Lucas sequences are defined by $F_n=F_{n-1}+F_{n-2}$ and $L_n=L_{n-1}+L_{n-2}$, where $F_0=0$, $F_1=1$, and $L_0=2$, $L_1=1$, respectively. Also Fibonacci and Lucas numbers for negative subscripts are defined as $F_{-n}=(-1)^{n+1}F_n$ and $L_{-n}=(-1)^nL_n$ for $n\geq 1$. For more information about Fibonacci and Lucas sequences we refer the reader to [9, 15, 20, 25].

2. Preliminaries

In this section, we give some theorems and identities to be used in the proofs of the main theorems. The following theorem is given in [6].

Theorem 1. *Let $k\geq 0$ be an integer. Then all nonnegative integer solutions of the equation $u^2-5v^2=4.5^k$ are given by*

$$(u,v)=\begin{cases}(5^{(k+1)/2}F_{2m+1},5^{(k-1)/2}L_{2m+1}), & k \text{ is an odd integer}\\ (5^{k/2}L_{2m},5^{k/2}F_{2m}), & k \text{ is an even integer}\end{cases}$$

and all nonnegative integer solutions of the equation $u^2-5v^2=-4.5^k$ are given by

$$(u,v)=\begin{cases}(5^{(k+1)/2}F_{2m},5^{(k-1)/2}L_{2m}), & k \text{ is an odd integer}\\ (5^{k/2}L_{2m+1},5^{k/2}F_{2m+1}), & k \text{ is an even integer}\end{cases}$$

with $m\geq 0$.

Now we give the following two theorems from [14].

Theorem 2. *Let $m>3$ be an integer and $F_n=F_mx^2$ for some integer x. Then $n=m$.*

Theorem 3. *Let $m\geq 2$ be an integer and $L_n=L_mx^2$ for some integer x. Then $n=m$.*

Now we recall some divisibility properties of Fibonacci and Lucas numbers. These properties are given in several sources such as [1, 15, 25]. Also one can find the proofs of the following theorems in [1, 13].

Theorem 4. *Let $m,n\in\mathbb{Z}$ and $n\geq 3$. Then $F_n|F_m$ if and only if $n|m$.*

Theorem 5. *Let $m,n\in\mathbb{Z}$ and $n\geq 2$. Then $L_n|F_m$ if and only if $n|m$ and m/n is an even integer.*

Theorem 6. *Let $m,n \in \mathbb{Z}$ and $n \geq 2$. Then $L_n | L_m$ if and only if $n|m$ and m/n is an odd integer.*

The following theorem is proved by Cohn in [2].

Theorem 7. *Assume that $n \geq 1$. If $F_n = x^2$, then $n = 1,2,12$. If $F_n = 2x^2$, then $n = 3$, 6. If $L_n = x^2$, then $n = 1,3$, and if $L_n = 2x^2$, then $n = 6$.*

Now we give some well known identities involving Fibonacci and Lucas numbers. The identities are as follows:

$$L_m F_n - F_m L_n = 2(-1)^m F_{n-m}, \tag{2.1}$$

$$L_m L_n - 5F_m F_n = 2(-1)^m L_{n-m}, \tag{2.2}$$

$$L_m L_n + 5F_m F_n = 2L_{n+m}, \tag{2.3}$$

$$L_m F_n + F_m L_n = 2F_{n+m}, \tag{2.4}$$

$$L_{3n} = L_n(L_n^2 - 3(-1)^n), \tag{2.5}$$

$$F_{2n} = F_n L_n, \tag{2.6}$$

$$2|F_n \iff 2|L_n \iff 3|n, \tag{2.7}$$

$$(F_n, L_n) = 1 \text{ or } (F_n, L_n) = 2, \tag{2.8}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{2.9}$$

$$F_{n+m}^2 - L_n F_{n+m} F_m + (-1)^n F_m^2 = (-1)^m F_n^2, \tag{2.10}$$

$$L_{n+m}^2 - 5F_n L_{n+m} F_m - 5(-1)^n F_m^2 = (-1)^m L_n^2, \tag{2.11}$$

and

$$5 \nmid L_n \tag{2.12}$$

for all $m,n \in \mathbb{Z}$.

The proof of the following theorem is given in [14].

Theorem 8. *There is no integer x such that $L_n = 2L_m x^2$ for $m > 1$.*

Before considering the main theorems of the paper, several theorems, which will be useful during the proofs of the main theorems, are needed.

Theorem 9. *Let $m > 1$ be an integer and $F_n = 2L_m x^2$. Then $m = 3$, $x^2 = 1$, and $n = 6$ or $m = 6$, $x^2 = 4$, and $n = 12$.*

Proof. Assume that $m > 1$ and $F_n = 2L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = 2L_m x^2$$

by (2.6).

Firstly, assume that k is an even integer. Then we can write $(F_{mk}/L_m)L_{mk} = 2x^2$. It can be seen easily that if $(F_{mk}/L_m, L_{mk}) = d$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$\frac{F_{mk}}{L_m} = u^2,\ L_{mk} = 2v^2 \tag{2.13}$$

or

$$\frac{F_{mk}}{L_m} = 2u^2,\ L_{mk} = v^2 \tag{2.14}$$

for some integers u and v with $\gcd(u,v) = 1$. Assume that (2.13) is satisfied. Then $mk = 6$ by Theorem 7. Since $m > 1$ and k is an even integer, we get $m = 3$ and $k = 2$. But this is impossible, since $u^2 = F_{mk}/L_m = F_6/L_3 = 2$. Assume that (2.14) is satisfied. Then $mk = 1$ or 3 by Theorem 7. But this contradicts the fact that k is an even integer. Assume that $d = 2$. Then

$$\frac{F_{mk}}{L_m} = 2u^2,\ L_{mk} = (2v)^2 \tag{2.15}$$

or

$$\frac{F_{mk}}{L_m} = (2u)^2,\ L_{mk} = 2v^2 \tag{2.16}$$

for some integers u and v with $\gcd(u,v) = 1$. A similar argument shows that (2.15) and (2.16) are impossible.

Secondly, assume that k is an odd integer. Then we can write $F_{mk}(L_{mk}/L_m) = 2x^2$. It can be easily seen that if $d = (F_{mk}, L_{mk}/L_m)$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$F_{mk} = u^2,\ \frac{L_{mk}}{L_m} = 2v^2 \tag{2.17}$$

or

$$F_{mk} = 2u^2,\ \frac{L_{mk}}{L_m} = v^2 \tag{2.18}$$

for some integers u and v with $\gcd(u,v) = 1$. It can be seen that the identity (2.17) is impossible by Theorem 8. Assume that (2.18) is satisfied. Then $mk = m$, i.e., $k = 1$ by Theorem 3. Therefore $F_m = 2u^2$ and this shows that $m = 3$ or 6 by Theorem 7. If $m = 3$ or 6, then it can be seen that $x^2 = 1$ and $n = 6$ or $x^2 = 4$ and $n = 12$, respectively. Assume that $d = 2$. Then

$$F_{mk} = 2u^2,\ \frac{L_{mk}}{L_m} = (2v)^2 \tag{2.19}$$

or

$$F_{mk} = (2u)^2,\ \frac{L_{mk}}{L_m} = 2v^2 \tag{2.20}$$

for some integers u and v with $\gcd(u,v) = 1$. The identity (2.19) is impossible by Theorem 3. The identity (2.20) is impossible by Theorem 8. This completes the proof. $\square$

Theorem 10. *Let $m \geq 2$ and $F_n = L_m x^2$. Then $m = 2$, $x^2 = 1$, and $n = 4$ or $m = 3$, $x^2 = 36$, and $n = 12$ or $m = 12$, $x^2 = 144$, and $n = 24$.*

Proof. Assume that $m \geq 2$ and $F_n = L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = L_m x^2$$

by (2.6).

Firstly, assume that k is an odd integer. Then $F_{mk}(L_{mk}/L_m) = x^2$. It can be easily shown that if $d = (F_{mk}, L_{mk}/L_m)$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then we get $F_{mk} = u^2$, $L_{mk}/L_m = v^2$ for some integers u and v with $\gcd(u, v) = 1$. Since $m \geq 2$, we get $mk = 2, 12$ by Theorem 7. If $mk = 2$, then $m = 2$, $k = 1$ and therefore we get $n = 4$ and $x^2 = 1$. If $mk = 12$, then $m = 12$, $k = 1$ or $m = 4$, $k = 3$. If $m = 12$ and $k = 1$, then $n = 24$ and $x^2 = 144$. If $m = 4$ and $k = 3$, then $v^2 = L_{mk}/L_m = L_{12}/L_4 = 46$, which is impossible. Now assume that $d = 2$. Then it is easily seen that

$$F_{mk} = 2u^2, \ \frac{L_{mk}}{L_m} = 2v^2 \tag{2.21}$$

for some integers u and v with $\gcd(u, v) = 1$. Since $L_{mk} = 2L_m v^2$ has no integer solution v for $m \geq 2$ by Theorem 8, the identity (2.21) is impossible.

Secondly, assume that k is an even integer. Then $(F_{mk}/L_m)L_{mk} = x^2$. It can be easily seen that if $d = (F_{mk}/L_m, L_{mk})$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then $F_{mk}/L_m = u^2$ and $L_{mk} = v^2$ for some positive integers u and v. Since $m \geq 2$, we get $mk = 3$ by Theorem 7. But this contradicts the fact that k is an even integer. Now assume that $d = 2$. Then it is easily seen that

$$\frac{F_{mk}}{L_m} = 2u^2, \ L_{mk} = 2v^2 \tag{2.22}$$

for some integers u and v with $\gcd(u, v) = 1$. By Theorem 7, it follows that $mk = 6$ and therefore $k = 2$, $m = 3$. If $k = 2$ and $m = 3$, then we get $n = 12$ and $x^2 = 36$. This completes the proof. □

Theorem 11. *Let $r > 3$ and F_r be a prime number. If $F_n = F_r L_m x^2$ with $m \geq 2$, then $m = 2$, $n = 12$, $r = 4$, and $x^2 = 16$ or $m = r$, $n = 2m$, and $x^2 = 1$.*

Proof. Assume that $F_n = F_r L_m x^2$. Then $L_m | F_n$ and therefore $n = 2mk$ for some natural number k by Theorem 5. Thus we get

$$F_n = F_{2mk} = F_{mk} L_{mk} = F_r L_m x^2$$

by (2.6).

Firstly, assume that k is an odd integer. Then $F_{mk}(L_{mk}/L_m) = F_r x^2$. It can be seen easily that if $(F_{mk}, L_{mk}/L_m) = d$, then $d = 1$ or $d = 2$ by (2.8). Assume that

$d = 1$. Since F_r is a prime number, we get

$$F_{mk} = u^2, \ \frac{L_{mk}}{L_m} = F_r v^2 \tag{2.23}$$

or

$$F_{mk} = F_r u^2, \ \frac{L_{mk}}{L_m} = v^2 \tag{2.24}$$

for some integers u and v. Assume that (2.23) is satisfied. Then $mk = 2$ or $mk = 12$ by Theorem 7. If $mk = 2$, then $m = 2$, $k = 1$, and therefore $F_r v^2 = L_{mk}/L_m = 1$, which is impossible. If $mk = 12$, then $m = 12$, $k = 1$ or $m = 4$, $k = 3$. A similar argument shows that $m = 12$, $k = 1$ and $m = 4$, $k = 3$ are impossible. Assume that (2.24) is satisfied. Then by Theorem 2 and Theorem 3, we get $mk = m = r$. This shows that $n = 2m$, and $x^2 = 1$ by Theorem 2. Assume that $d = 2$. Then, since F_r is odd prime number, it is seen that

$$F_{mk} = 2u^2, \ \frac{L_{mk}}{L_m} = 2F_r v^2 \tag{2.25}$$

or

$$F_{mk} = 2F_r u^2, \ \frac{L_{mk}}{L_m} = 2v^2 \tag{2.26}$$

for some integers u and v. Assume that (2.25) is satisfied. Then $mk = 3$ or $mk = 6$ by Theorem 7. If $mk = 3$, then $m = 3$, $k = 1$. If $m = 3$ and $k = 1$, then $F_r v^2 = L_{mk}/2L_m = 1/2$, which is impossible. If $mk = 6$, then $m = 6$, $k = 1$ or $m = 2$, $k = 3$. If $m = 6$, $k = 1$, then $F_r v^2 = L_{mk}/2L_m = 1/2$, which is impossible. If $k = 3$ and $m = 2$, then it can be seen that $n = 12$, $r = 4$, $x^2 = 16$. The identity (2.26) is impossible by Theorem 8.

Secondly, assume that k is an even integer. Then $(F_{mk}/L_m)L_{mk} = F_r x^2$. It can be easily seen that if $d = (F_{mk}/L_m, L_{mk})$, then $d = 1$ or $d = 2$ by (2.8). Assume that $d = 1$. Then

$$\frac{F_{mk}}{L_m} = u^2, \ L_{mk} = F_r v^2 \tag{2.27}$$

or

$$\frac{F_{mk}}{L_m} = F_r u^2, \ L_{mk} = v^2 \tag{2.28}$$

for some integers u and v. Assume that (2.27) is satisfied. Then by Theorem 10, it follows that $m = 2$, $k = 2$ or $m = 3$, $k = 4$ or $m = 12$, $k = 2$. But there is no solution of the equation $L_{mk} = F_r v^2$ in all situations. Assume that (2.28) is satisfied. Then $mk = 1$ or 3 by Theorem 7. But this is impossible since k is an even integer. Now assume that $d = 2$. Then we get

$$\frac{F_{mk}}{L_m} = 2u^2, \ L_{mk} = 2F_r v^2 \tag{2.29}$$

or

$$\frac{F_{mk}}{L_m} = 2F_r u^2, \ L_{mk} = 2v^2 \tag{2.30}$$

for some integers u and v. Assume that (2.29) is satisfied. Then $m = 3$, $k = 2$ or $m = 6$, $k = 2$ by Theorem 9. But there is no solution of the equation $L_{mk} = 2F_r v^2$ when $m = 3$, $k = 2$ or $m = 6$, $k = 2$. Assume that (2.30) is satisfied. Then $mk = 6$ by Theorem 7. It follows that $m = 3$, $k = 2$ and therefore $F_r u^2 = F_{mk}/2L_m = F_6/2L_3 = 1$. But this is impossible since F_r is a prime number. This completes the proof. $\square$

3. Main Theorems

In this section, we determine when the Diophantine equations

$$x^2 - 5F_n xy - 5(-1)^n y^2 = 5^r \tag{3.1}$$

and

$$x^2 - 5F_n xy - 5(-1)^n y^2 = -5^r \tag{3.2}$$

have positive integer solutions under the assumptions that $n \geq 1$, $r \geq 0$. After then, we find all positive integer solutions of the related Diophantine equations. Firstly, let us consider the above equations for $n = 1$. Then (3.1) and (3.2) turns into the equations

$$x^2 - 5xy + 5y^2 = 5^r$$

and

$$x^2 - 5xy + 5y^2 = -5^r,$$

respectively. Multiplying both sides of the above two equations by 4 and completing the square reduce them to the equations $u^2 - 5v^2 = 4.5^r$ and $u^2 - 5v^2 = -4.5^r$, respectively. So by means of Theorem 1, we can easily find all positive integer solutions of the equations $x^2 - 5xy + 5y^2 = 5^r$ and $x^2 - 5xy + 5y^2 = -5^r$. Now we will give all positive integer solutions of the equations $x^2 - 5xy + 5y^2 = 5^r$ and $x^2 - 5xy + 5y^2 = -5^r$ for the sake of completeness.

Theorem 12. *Let $r \geq 0$ be an integer. If r is an even integer, then all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = 5^r$ are given by $(x, y) = (5^{r/2}L_{2m+1}, 5^{r/2}F_{2m})$ or $(x, y) = (5^{r/2}L_{2m-1}, 5^{r/2}F_{2m})$ with $m \geq 1$, and all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = -5^r$ are given by $(x, y) = (5^{r/2}L_{2m+2}, 5^{r/2}F_{2m+1})$ or $(x, y) = (5^{r/2}L_{2m}, 5^{r/2}F_{2m+1})$ with $m \geq 0$. If r is an odd integer, then all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = 5^r$ are given by $(x, y) = (5^{(r+1)/2}F_{2m+2}, 5^{(r-1)/2}L_{2m+1})$ with $m \geq 0$ or $(x, y) = (5^{(r+1)/2}F_{2m}, 5^{(r-1)/2}L_{2m+1})$ with $m \geq 1$, and all positive integer solutions of the equation $x^2 - 5xy + 5y^2 = -5^r$ are given by $(x, y) = (5^{(r+1)/2}F_{2m+1}, 5^{(r-1)/2}L_{2m})$ or $(x, y) = (5^{(r+1)/2}F_{2m-1}, 5^{(r-1)/2}L_{2m})$ with $m \geq 0$.*

From now on, we will assume that $n > 1$ and $k \geq 0$.

Theorem 13. *If n is an even integer, then all positive integer solutions of the equation $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$ are given by*

$$(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$$

with $m\geq 1$ and $n|m$, and if n is an odd integer, then all positive integer solutions of the equation $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$ are given by $(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$ or $(x,y)=(5^kL_{2m-n}/L_n,5^kF_{2m}/L_n)$ with $m\geq 1$ and $n|m$.

Proof. Assume that $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$ for some positive integers x and y. Multiplying both sides of the equation by 4 and then completing the square give

$$(2x-5F_ny)^2-\left(25F_n^2+20(-1)^n\right)y^2=4.5^{2k}. \tag{3.3}$$

It is easily seen from (2.9) that $25F_n^2+20(-1)^n=5L_n^2$. Thus the equation (3.3) becomes $(2x-5F_ny)^2-5(L_ny)^2=4.5^{2k}$. By Theorem 1, we obtain $|2x-5F_ny|=5^kL_{2m}$ and $L_ny=5^kF_{2m}$. Since $5\nmid L_n$ and $L_ny=5^kF_{2m}$, it follows that $L_n|F_{2m}$. Thus by Theorem 5, $n|2m$ and $2m/n$ is an even integer. This shows that $n|m$. Then from the equality $L_ny=5^kF_{2m}$, we obtain $y=5^kF_{2m}/L_n$. Assume that $2x-5F_ny=5^kL_{2m}$. Then substituting the value of y into $2x-5F_ny=5^kL_{2m}$, we obtain

$$x=\frac{5^k(L_{2m}L_n+5F_nF_{2m})}{2L_n}.$$

By using the identity (2.3), we get $x=5^kL_{2m+n}/L_n$. Now assume that $2x-5F_ny=-5^kL_{2m}$. Then substituting the value of y into $2x-5F_ny=-5^kL_{2m}$ and using the identity (2.2), we obtain $x=-5^kL_{n-2m}/L_n$. If n is even, then x is negative and we omit it. So n is odd and therefore $x=5^kL_{2m-n}/L_n$. Thus when n is even, all positive integer solutions of the equation $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$ are given by $(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$ with $m\geq 1$, and when n is odd, all positive integer solutions of the equation $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$ are given by $(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$ or $(x,y)=(5^kL_{2m-n}/L_n,5^kF_{2m}/L_n)$ with $m\geq 1$. Conversely, if n is an even integer and $(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$ with $m\geq 1$ and $n|m$, and if n is an odd integer and $(x,y)=(5^kL_{2m+n}/L_n,5^kF_{2m}/L_n)$ or $(x,y)=(5^kL_{2m-n}/L_n,5^kF_{2m}/L_n)$ with $m\geq 1$ and $n|m$, then by (2.11), it follows that $x^2-5F_nxy-5(-1)^ny^2=5^{2k}$. $\square$

Theorem 14. *The equation $x^2-5F_nxy-5(-1)^ny^2=-5^{2k}$ has no positive integer solutions x and y.*

Proof. Assume that $x^2-5F_nxy-5(-1)^ny^2=-5^{2k}$ for some positive integers x and y. Multiplying both sides of the equation by 4 and then completing the square give

$$(2x-5F_ny)^2-\left(25F_n^2+20(-1)^n\right)y^2=-4.5^{2k}. \tag{3.4}$$

If we use the fact that $25F_n^2 + 20(-1)^n = 5L_n^2$ from (2.9), the equation (3.4) becomes $(2x - 5F_n y)^2 - 5(L_n y)^2 = -4.5^{2k}$. By Theorem 1, we obtain $|2x - 5F_n y| = 5^k L_{2m+1}$ and $L_n y = 5^k F_{2m+1}$. Since $5 \nmid L_n$ and $L_n y = 5^k F_{2m+1}$, it follows that $L_n | F_{2m+1}$. Thus by Theorem 5, $n|2m+1$ and $(2m+1)/n$ is an even integer, which is impossible. This completes the proof. $\square$

Theorem 15. *If n is an even integer, then the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ has no positive integer solutions x and y. If n is an odd integer, then all positive integer solutions of the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ are given by $(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n|2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n|2m+1$.*

Proof. Assume that $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$ for some positive integers x and y. Multiplying both sides of the equation by 4, we obtain $4x^2 - 20F_n xy - 20(-1)^n y^2 = 4.5^{2k+1}$. Next completing the square and using the fact that $25F_n^2 + 20(-1)^n = 5L_n^2$ give

$$(2x - 5F_n y)^2 - 5(L_n y)^2 = 4.5^{2k+1}.$$

By Theorem 1, we obtain $|2x - 5F_n y| = 5^{k+1} F_{2m+1}$ and $L_n y = 5^k L_{2m+1}$. Since $5 \nmid L_n$ and $L_n y = 5^k L_{2m+1}$, it follows that $L_n | L_{2m+1}$, which implies that $n|2m+1$ and $(2m+1)/n$ is an odd integer by Theorem 6. This shows that n must be an odd integer. Then from the equality $L_n y = 5^k L_{2m+1}$, we obtain $y = 5^k L_{2m+1}/L_n$. Substituting the value of y into the equality $|2x - 5F_n y| = 5^{k+1} F_{2m+1}$ and using (2.1) and (2.4) give $x = 5^{k+1} F_{2m+1+n}/L_n$ or $x = 5^{k+1} F_{2m+1-n}/L_n$. So $(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n|2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n|2m+1$. Conversely, if n is an odd integer and

$$(x, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$$

with $m \geq 1$ and $n \mid 2m+1$ or $(x, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n|2m+1$, then by (2.9) and (2.10), it follows that $x^2 - 5F_n xy - 5(-1)^n y^2 = 5^{2k+1}$. This completes the proof. $\square$

Theorem 16. *If n is an odd integer, then the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ has no positive integer solutions x and y. If n is an even integer, then all positive integer solutions of the equation $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ are given by $(x, y) = (5^{k+1} F_{nt+n}/L_n, 5^k L_{nt}/L_n)$, where t is an odd integer.*

Proof. Assume that $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$ for some positive integers x and y. Multiplying both sides of the equation by 4 and then completing the square give

$$(2x - 5F_n y)^2 - 5(L_n y)^2 = -4.5^{2k+1}.$$

By Theorem 1, we obtain $|2x - 5F_n y| = 5^{k+1}F_{2m}$ and $L_n y = 5^k L_{2m}$. Since $5 \nmid L_n$ and $L_n y = 5^k L_{2m}$, it follows that $L_n | L_{2m}$, which implies that $n|2m$ and $2m/n$ is an odd integer by Theorem 6. Then $2m = nt$ for some odd integer t. This shows that n must be even. Then from the equality $L_n y = 5^k L_{2m}$, we obtain $y = 5^k L_{nt}/L_n$. Substituting the value of y into the equality $|2x - 5F_n y| = 5^{k+1}F_{2m}$ and using (2.1) and (2.4) give $x = 5^{k+1}F_{nt+n}/L_n$ or $x = 5^{k+1}F_{n-nt}/L_n$. Since $n|nt$, it follows that $n \leq nt$ and so the solution $x = 5^{k+1}F_{n-nt}/L_n$ is zero or negative. Therefore we omit it. Thus

$$(x,y) = (5^{k+1}F_{nt+n}/L_n, 5^k L_{nt}/L_n),$$

where t is an odd integer. Conversely, if n is an even integer and $(x,y) = (5^{k+1}F_{nt+n}/L_n, 5^k L_{nt}/L_n)$, where t is an odd integer, then by (2.9) and (2.10), it follows that $x^2 - 5F_n xy - 5(-1)^n y^2 = -5^{2k+1}$. □

Theorem 17. *If k is an even integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ has positive integer solutions x and y only when $n = 2, 3, 12$. Moreover, the only positive integer solution of the the equation $x^2 - 5F_2 xy^2 - 5y^4 = 5^{2k}$ is given by $(x,y) = (6.5^k, 5^{k/2})$, all positive integer solutions of the equation $x^2 - 5F_3 xy^2 + 5y^4 = 5^{2k}$ are given by $(x,y) = (19.5^k, 6.5^{k/2})$ or $(x,y) = (341.5^k, 6.5^{k/2})$, and the only positive integer solution of the the equation $x^2 - 5F_{12} xy^2 - 5y^4 = 5^{2k}$ is given by $(x,y) = (103681.5^k, 12.5^{k/2})$. If k is an odd integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ has positive integer solutions x and y only when $n = 5$, in which case all positive integer solutions are given by $(x,y) = (5^k, 5^{(k+1)/2})$ or $(x,y) = (124.5^k, 5^{(k+1)/2})$.*

Proof. Assume that $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k}$ for some positive integers x and y. By Theorem 13, if n is even, then it follows that

$$(x, y^2) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n) \tag{3.5}$$

with $m \geq 1$ and $n|m$, and if n is odd, then we clearly have that

$$(x, y^2) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n) \tag{3.6}$$

or

$$(x, y^2) = (5^k L_{2m-n}/L_n, 5^k F_{2m}/L_n) \tag{3.7}$$

with $m \geq 1$ and $n|m$. Thus we obtain $y^2 = 5^k F_{2m}/L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then it follows that $F_{2m} = L_n u^2$ for some integer u. By Theorem 10, it is seen that $n = 2$, $m = 2$ or $n = 3$, $m = 6$ or $n = 12$, $m = 12$. Substituting these values of n and m into (3.5), (3.6), and (3.7), we obtain $(x,y) = (6.5^k, 5^{k/2})$ or $(x,y) = (19.5^k, 6.5^{k/2})$ or $(x,y) = (341.5^k, 6.5^{k/2})$ or $(x,y) = (103681.5^k, 12.5^{k/2})$, respectively.

Case 2: Assume that k is an odd integer. Then it follows that $L_n = 5F_{2m}v^2 = F_5 F_{2m} v^2$ for some integer v. By Theorem 11, it is seen that $n = 5$ and $m = 5$.

Substituting these values of n and m into (3.6), we obtain $(x,y) = (5^k, 5^{(k+1)/2})$ or $(x,y) = (124.5^k, 5^{(k+1)/2})$. This completes the proof. $\square$

Theorem 18. *Assume that n is an odd integer. If k is an odd integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k+1}$ has no positive integer solutions x and y. If k is an even integer, then there is only one positive integer solution of the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k+1}$ given by $(x,y) = (5^{k+1}F_n, 5^{k/2})$.*

Proof. Assume that $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = 5^{2k+1}$ for some positive integers x and y. By Theorem 15, if n is an odd integer, then it follows that $(x,y^2) = (5^{k+1}F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n|2m+1$ or $(x,y^2) = (5^{k+1}F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n|2m+1$. Thus we obtain $y^2 = 5^k L_{2m+1}/L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then from the equality $y^2 = 5^k L_{2m+1}/L_n$, it clearly follows that $L_{2m+1} = L_n u^2$ for some integer u. By Theorem 3, it is seen that $n = 2m+1$ and therefore $(x,y) = (5^{k+1}F_n, 5^{k/2})$.

Case 2: Assume that k is an odd integer. Since $y^2 = 5^k L_{2m+1}/L_n$, we clearly have that $L_{2m+1} = 5L_n v^2$ for some integer v. But this is impossible since $5 \nmid L_n$. This completes the proof. $\square$

Theorem 19. *Assume that n is an even integer. If k is an odd integer, then the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = -5^{2k+1}$ has no positive integer solutions x and y. If k is an even integer, then there is only one positive integer solution of the equation $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = -5^{2k+1}$ given by $(x,y) = (5^{k+1}F_n, 5^{k/2})$.*

Proof. Assume that $x^2 - 5F_n xy^2 - 5(-1)^n y^4 = -5^{2k+1}$ for some positive integers x and y. By Theorem 16, if n is an even integer, then it follows that $(x,y^2) = (5^{k+1}F_{nt+n}/L_n, 5^k L_{nt}/L_n)$, where t is an odd integer. Thus $y^2 = 5^k L_{nt}/L_n$. Now we divide the proof into two cases.

Case 1: Assume that k is an even integer. Then it follows that $L_{nt} = L_n u^2$ for some integer u. By Theorem 3, it is seen that $nt = n$ and therefore $t = 1$. Thus we obtain $(x,y) = (5^{k+1}F_n, 5^{k/2})$.

Case 2: Assume that k is an odd integer. Then it follows that $L_{nt} = 5L_n v^2$ for some integer v. But this is impossible since $5 \nmid L_n$. This completes the proof. $\square$

Theorem 20. *The equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k}$ has positive integer solutions x and y only when k is even and n is odd, in which case there is only one positive integer solution given by $(x,y) = (5^{k/2}, 5^k F_n)$.*

Proof. Assume that $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k}$ for some positive integers x and y. By Theorem 13, if n is an even integer, then it follows that $(x^2,y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$ and $n|m$, and if n is an odd integer, then it follows that $(x^2,y) = (5^k L_{2m+n}/L_n, 5^k F_{2m}/L_n)$ or $(x^2,y) = (5^k L_{2m-n}/L_n, 5^k F_{2m}/L_n)$ with $m \geq 1$ and $n|m$. From now on, we divide the proof into two cases.

Case 1: Assume that n is an even integer. Then it follows that $x^2 = 5^k L_{2m+n}/L_n$. Here if k is an even integer, then we get $L_{2m+n} = L_n u^2$ for some integer u. By Theorem 3, it is seen that $n = n+2m$ and thus $m = 0$, which contradicts the fact that $m \geq 2$. If k is an odd integer, then a simple computation shows that $L_{2m+n} = 5L_n v^2$ for some integer v, which is impossible since $5 \nmid L_n$.

Case 2: Assume that n is an odd integer. Here we are not interested in the solution $x^2 = 5^k L_{2m+n}/L_n$. Because it is clear from Case 1 that, we have no solutions. Thus it follows that $x^2 = 5^k L_{2m-n}/L_n$. Here if k is an even integer, then we get $L_{2m-n} = L_n u^2$ for some integer u. By Theorem 3, it is seen that $2m-n = n$ and therefore $m = n$. Thus the only one positive integer solution is given by $(x,y) = (5^{k/2}, 5^k F_n)$. If k is an odd integer, then a simple computation shows that $L_{2m-n} = 5L_n v^2$ for some integer v, which is impossible since $5 \nmid L_n$. This completes the proof. □

Theorem 21. *Assume that n is odd. If k is an even integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ has positive integer solutions x and y only when $n = 5$, in which case all positive integer solutions are given by $(x,y) = (5^{(k+2)/2}, 5^k)$ or $(x,y) = (5^{(k+2)/2}, 124.5^k)$. If k is an odd integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ has positive integer solutions x and y only when $n = 3$, in which case all positive integer solutions are given by $(x,y) = (6.5^{(k+1)/2}, 19.5^k)$ or $(x,y) = (6.5^{(k+1)/2}, 341.5^k)$.*

Proof. Assume that $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = 5^{2k+1}$ for some positive integers x and y. By Theorem 15, if n is an odd integer, then it follows that $(x^2, y) = (5^{k+1} F_{2m+1+n}/L_n, 5^k L_{2m+1}/L_n)$ with $m \geq 1$ and $n | 2m+1$ or $(x^2, y) = (5^{k+1} F_{2m+1-n}/L_n, 5^k L_{2m+1}/L_n)$ with $m > 1$ and $n | 2m+1$. Assume that $x^2 = 5^{k+1} F_{2m+1+n}/L_n$. If k is even, then we get $F_{2m+n+1} = 5L_n u^2$ for some integer u. Since n is odd, it follows that $n = 5$, $m = 2$ by Theorem 11. Thus we obtain $(x,y) = (5^{(k+2)/2}, 5^k)$. If k is odd, then we get $F_{2m+n+1} = L_n v^2$ for some integer v. Since n is odd, it is seen that $n = 3$, $m = 4$ by Theorem 10. Thus we obtain $(x,y) = (6.5^{(k+1)/2}, 19.5^k)$. Now assume that $x^2 = 5^{k+1} F_{2m+1-n}/L_n$. If k is even, then we get $F_{2m+1-n} = 5L_n u^2 = F_5 L_n u^2$ for some integer u. Since n is odd, this shows that $n = 5$, $m = 7$ by Theorem 11. Thus we obtain $(x,y) = (5^{(k+2)/2}, 124.5^k)$. If k is odd, then it follows that $F_{2m+1-n} = L_n v^2$ for some integer v. Since n is odd, it is seen that $n = 3$, $m = 7$ by Theorem 10. Thus we obtain $(x,y) = (6.5^{(k+1)/2}, 341.5^k)$. This completes the proof. □

Since the proof of the following theorem is similar to that of the above theorem, we omit it.

Theorem 22. *Assume that n is even. If k is an even integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = -5^{2k+1}$ has no positive integer solutions x and y. If k is an odd integer, then the equation $x^4 - 5F_n x^2 y - 5(-1)^n y^2 = -5^{2k+1}$ has positive* *integer solutions x and y only when $n = 2, 12$, in which cases there is only one solution given by $(x, y) = (5^{(k+1)/2}, 5^k)$ and $(x, y) = (12.5^{(k+1)/2}, 5^k)$, respectively.*

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